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INST FOR PHYSICAL SCIENCE AND TECH. L R SCOTT ET AL.

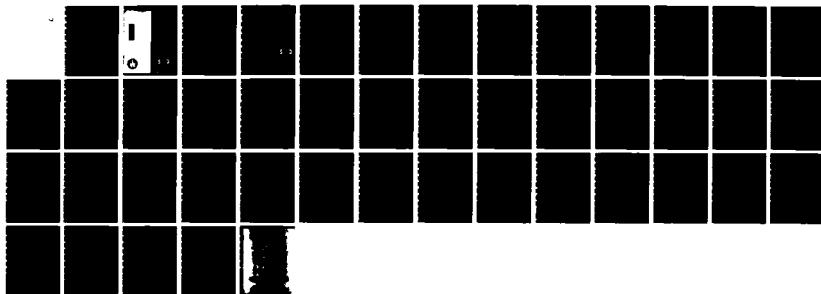
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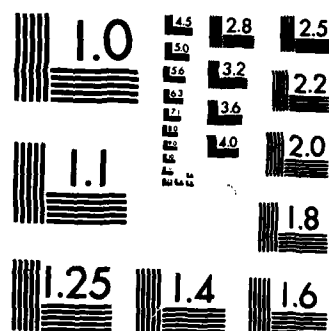
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by

L. R. Scott

and

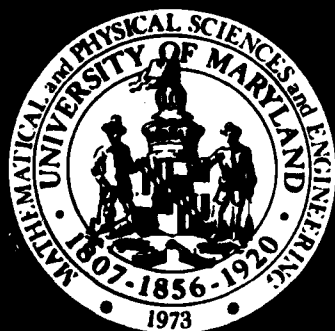
M. Vogelius

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Conforming finite element methods for incompressible  
and nearly incompressible continua

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# 1. Introduction

We shall be interested here <sup>is</sup> in finite element discretizations of problems involving an incompressibility condition. As model problems we consider the Stokes equations for the flow of a viscous, incompressible fluid and the equations of linear plane-strain elasticity for the deformation of an isotropic, nearly incompressible solid. In both cases the incompressibility condition takes the form of a divergence constraint. Although this is the most simple formulation, the proper understanding of how an approximate method satisfies the constraint represents an important step towards the understanding of more complicated situations, involving e.g. the Navier-Stokes equations or the equations of nonlinear elasticity. The finite element methods we study have the property that the approximations to the velocities, respectively to the displacements, are continuous; such methods are generally referred to as conforming.



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## 2. The Stokes equations

Let  $\Omega$  be a bounded polygonal domain in the plane, and let  $\underline{U}$  and  $P$  solve

$$(2.1) \quad \begin{cases} -\Delta \underline{U} + \nabla P = \underline{F} & \text{in } \Omega, \\ \nabla \cdot \underline{U} = 0 & \text{in } \Omega, \text{ and} \\ \underline{U} = 0 & \text{on } \partial\Omega. \end{cases}$$

Here  $\underline{U} = (U_1, U_2)$  represents the fluid velocities and  $P$  the pressure; the viscosity has been set to 1. To simplify the exposition we are assuming homogeneous boundary data on  $\partial\Omega$ . For the linear problem (2.1) we can convert inhomogeneous boundary data into an external force term  $\underline{F}$ . However, for a nonlinear problem one must deal directly with the inhomogeneous boundary data, cf. Gunzburger & Peterson [17]. Other possible boundary conditions could involve the normal fluid stresses

$$\sum_{j=1}^2 \left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) n_j - P n_i, \quad i = 1, 2,$$

but since stress boundary conditions are physically much more frequent when dealing with solids, we shall reserve these for our formulation of the boundary value problem for the equations of elasticity. For regularity results concerning the solution to (2.1) on a polygonal domain see Kellogg & Osborn [20] and Osborn [29]. Note that  $P$  is determined only up to an additive constant.

### 3. The equations of linear elasticity

As before let  $\Omega$  be a bounded polygonal domain in the plane and consider the problem

$$(3.1) \quad \begin{cases} -\Delta \underline{U} - \frac{1}{1-2\nu} \nabla(\nabla \cdot \underline{U}) = \underline{F} & \text{in } \Omega, \text{ and} \\ \sum_{j=1}^2 \epsilon_{ij}(\underline{U}) n_j + \frac{\nu}{1-2\nu} \nabla \cdot \underline{U} n_i = g_i & \text{on } \partial\Omega, \quad i = 1, 2, \end{cases}$$

$\underline{n} = (n_1, n_2)$  here denotes the outward unit normal to  $\Omega$ , and  $\epsilon_{ij}(\underline{U})$  is the usual symmetric strain tensor

$$\epsilon_{ij}(\underline{U}) = \frac{1}{2} \left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right).$$

The equations (3.1) are the equations of isotropic, plane-strain linear elasticity corresponding to the domain  $\Omega$ .  $0 < \nu < 1/2$  is a material-dependent constant, the so-called Poisson's ratio, which describes the compressibility.

The other constant, the shear modulus, that is needed in order to characterize fully an isotropic material has been absorbed into the external load  $\underline{F}$  and boundary load  $\underline{g}$ . If  $\underline{U}$  is a solution to (3.1) then the vector

$$\tilde{\underline{U}} = (U_1, U_2, 0)$$

solves the equations of 3-dimensional isotropic elasticity on the domain

$$\Omega \times \mathbb{R},$$

with external load  $\tilde{\underline{F}} = (\underline{F}, 0)$ , independent of  $x_3$ , and boundary load  $\tilde{\underline{g}} = (\underline{g}, 0)$ , independent of  $x_3$ , on the vertical boundary  $\partial\Omega \times \mathbb{R}$ . This is the reason for the notion of "plane-strain" in connection with (3.1). Values of  $\nu$  near  $1/2$  correspond to a nearly incompressible material.



Both the problem (2.1) and the problem (3.1) can (on a simply connected domain) be reduced to the solution of the biharmonic equation with Dirichlet boundary conditions. For (2.1) this follows through the introduction of a stream function such that  $\underline{U} = \nabla \times \psi$ ; the corresponding boundary conditions for  $\psi$  are homogeneous Dirichlet. For (3.1) one first subtracts a particular solution corresponding to the external load  $\underline{F}$  so that the resulting system has a vanishing external load. For such a system one may introduce an Airy stress function  $\phi$  ( $\Delta^2 \phi = 0$ ) so that

$$\left(\frac{\partial}{\partial x_1}\right)^2 \phi = \sigma_{22}, \quad -\frac{\partial^2}{\partial x_1 \partial x_2} \phi = \sigma_{12}, \quad \left(\frac{\partial}{\partial x_2}\right)^2 \phi = \sigma_{11},$$

where

$$\sigma_{ij} = \frac{E}{1+\nu} \left( \varepsilon_{ij}(\underline{U}) + \delta_{ij} \frac{\nu}{1-2\nu} \nabla \cdot \underline{U} \right)$$

denote the stresses corresponding to the displacement  $\underline{U}$  ( $E$  is the so-called Youngs modulus). The boundary conditions for  $\phi$  are in this case (inhomogeneous) Dirichlet.

Regularity results and a priori estimates for solutions to (3.1) (as well as (2.1)) may thus be derived from the properties of solutions to the biharmonic equation on a polygonal domain (cf. Grisvard [15]).

#### 4. Variational formulations

In order to introduce finite element discretizations of the equations (2.1) and (3.1) we have to cast them in a variational (or weak) form. This is done following the standard 3-step recipe:

- 1) multiply each differential equation by a suitable test function
- 2) integrate the result over  $\Omega$
- 3) integrate by parts (to taste).

By multiplication of the first equation in (2.1) by  $\underline{v}$  (vanishing on  $\partial\Omega$ ) the above three steps lead to

$$a(\underline{U}, \underline{v}) + b(\underline{v}, P) = (\underline{F}, \underline{v})$$

where  $(\underline{F}, \underline{v})$  is the usual  $[L_2(\Omega)]^2$  inner product,

$$(4.1) \quad a(\underline{U}, \underline{v}) = 2 \int_{\Omega} \sum_{i,j} \epsilon_{ij}(\underline{U}) \epsilon_{ij}(\underline{v}) d\mathbf{x} , \quad \text{and}$$

$$(4.2) \quad b(\underline{v}, P) = - \int_{\Omega} \nabla \cdot \underline{v} P d\mathbf{x} .$$

Multiplying the second equation in (2.1) by a suitable function  $q$  and integrating over  $\Omega$ , we get

$$b(\underline{U}, q) = 0 .$$

The appropriate spaces of "test" functions  $\underline{v}$  and  $q$  are given by

$$(4.3) \quad H^1_0(\Omega) \times H^1_0(\Omega) \quad \text{and} \quad L_2(\Omega) .$$

Here,  $H^1_0(\Omega)$  is the standard Sobolev space of functions whose gradients are square integrable and whose traces vanish on  $\partial\Omega$ . With these spaces our

"variational form" of (2.1) is

$$(4.4) \quad \left\{ \begin{array}{l} \text{Find } \underline{U} \in [H^1(\Omega)]^2 \text{ and } P \in L_2(\Omega) \text{ such that} \\ a(\underline{U}, \underline{v}) + b(\underline{v}, P) = (\underline{F}, \underline{v}) \text{ for all } \underline{v} \in [H^1(\Omega)]^2 \\ b(\underline{U}, q) = 0 \text{ for all } q \in L_2(\Omega) . \end{array} \right.$$

In order to find the variational formulation of (3.1) we multiply the first equation by  $\underline{v} \in [H^1(\Omega)]^2$  (not vanishing on  $\partial\Omega$ ) and integrate by parts. Because of the form of the boundary conditions this leads to

$$a(\underline{U}, \underline{v}) + b(\underline{v}, -\frac{2v}{1-2v} \nabla \cdot \underline{U}) = (\underline{F}, \underline{v}) + 2\langle \underline{g}, \underline{v} \rangle ,$$

where  $\langle \cdot, \cdot \rangle$  is the  $[L_2(\partial\Omega)]^2$  inner product. The differential equation (3.1) therefore has the weak form

$$(4.5) \quad \left\{ \begin{array}{l} \text{Find } \underline{U} \in [H^1(\Omega)]^2 \text{ such that} \\ a(\underline{U}, \underline{v}) + b(\underline{v}, -\frac{2v}{1-2v} \nabla \cdot \underline{U}) = (\underline{F}, \underline{v}) + 2\langle \underline{g}, \underline{v} \rangle . \end{array} \right.$$

This may be rewritten as

$$\left\{ \begin{array}{l} \text{Find } \underline{U} \in [H^1(\Omega)]^2 \text{ and } P \in L_2(\Omega) \text{ such that} \\ a(\underline{U}, \underline{v}) + b(\underline{v}, P) = (\underline{F}, \underline{v}) + 2\langle \underline{g}, \underline{v} \rangle \text{ for all } \underline{v} \in [H^1(\Omega)]^2 \\ -\frac{1-2v}{2v} (P, q) + b(\underline{U}, q) = 0 \text{ for all } q \in L_2(\Omega) . \end{array} \right.$$

Setting  $v = 1/2$  this gives "formally" a Stokes equation of the form (4.4) only, of course, with different boundary conditions. Indeed the equations of

elasticity for  $\nu$  near  $1/2$  may be viewed as a penalized version of a Stokes problem (for more details, see e.g. Temam [35]). Note that the equation (4.5) only has a solution provided the loads  $\underline{F}$  and  $\underline{g}$  are statically admissible, i.e. provided

$$(\underline{F}, \underline{R}) + 2\langle \underline{g}, \underline{R} \rangle = 0$$

for any rigid motion  $\underline{R}$ . The solution  $\underline{U}$  is also only determined modulo a rigid motion; whenever we discuss the problem (4.5) we shall avoid this non-uniqueness by thinking of functions as equivalence classes modulo rigid motions.

(remember, a rigid motion is one for which  $\epsilon_{ij}(\underline{R}) = 0$ , or  $\underline{R}(x_1, x_2) = (-\gamma x_2 + \alpha, \gamma x_1 + \beta)$ ). In this paper we shall always assume that  $\Omega$  is connected and for the Stokes equations we make  $P$  unique by imposing

$$\int_{\Omega} P \, d\underline{x} = 0.$$

This replaces  $L_2(\Omega)$  in (4.4) by  $\overset{\circ}{L}_2(\Omega) = L_2(\Omega) \cap \{q : \int_{\Omega} q \, d\underline{x} = 0\}$ .

### 5. Stability of finite element approximations

Given a variational formulations such as (4.4) or (4.5) the finite element method consists of choosing finite dimensional subspaces

$$\overset{\circ}{V}_h \subseteq [H^1(\Omega)]^2, \quad \Pi_h \subseteq \overset{\circ}{L}_2(\Omega)$$

or

$$V_h \subseteq [H^1(\Omega)]^2,$$

and replacing everywhere the infinite dimensional spaces by their finite dimensional counterparts.\* The spaces  $V_h$ ,  $\overset{\circ}{V}_h$  and  $\Pi_h$  are typically made up of piecewise polynomial functions on some triangulation  $\sum_h$ ;  $h$  denotes the mesh size. For the Stokes problem the discrete version thus becomes

$$(5.1) \quad \left\{ \begin{array}{l} \text{Find } \underline{U}_h \in \overset{\circ}{V}_h \text{ and } P_h \in \Pi_h \text{ such that} \\ a(\underline{U}_h, \underline{v}) + b(\underline{v}, P_h) = (\underline{F}, \underline{v}) \text{ for all } \underline{v} \in \overset{\circ}{V}_h \\ b(\underline{U}_h, q) = 0 \text{ for all } q \in \Pi_h \end{array} \right.$$

and for the equations of elasticity (4.5) it reads

$$(5.2) \quad \left\{ \begin{array}{l} \text{Find } \underline{U}_h \in V_h \text{ such that} \\ a(\underline{U}_h, \underline{v}) + b(\underline{v}, -\frac{2\nu}{1-2\nu} \nabla \cdot \underline{U}_h) = (\underline{F}, \underline{v}) + 2\langle \underline{g}, \underline{v} \rangle \\ \text{for all } \underline{v} \in V_h. \end{array} \right.$$

---

\* Indeed  $V_h$  should be a subset of  $[H^1(\Omega)]^2 / \{\text{Rigid motions}\}$  but this is not explicitly mentioned: our convention is to identify a function with its equivalence class modulo rigid motions, whenever appropriate.

As an example we consider a uniform triangulation which locally is as shown in Fig. 1. Perhaps the simplest finite dimensional subspaces are

$$\begin{aligned} V_h &= [C^0 \text{ piecewise linear functions}]^2, \text{ or} \\ (5.3) \quad \overset{\circ}{V}_h &= [C^0 \text{ piecewise linear functions that vanish on } \partial\Omega]^2 \\ &\text{and correspondingly} \end{aligned}$$

$$\Pi_h = \text{piecewise constants having integral zero (over } \Omega \text{)}.$$

One reason why finite element methods for the equations of elasticity (near incompressibility), or the Stokes equations, are intriguing is that the choices of spaces (5.3) do not work for the mesh in Fig. 1! In the case of the discrete equations of elasticity (5.2) the relative error as  $\nu \rightarrow 1/2$  for fixed  $h$  approaches a constant, which is bounded away from 0 independently of  $h$ . For the discrete Stokes equation (5.1),  $\underline{U}_h \equiv 0$  for any  $h$ . Below we give an explanation of this phenomenon in the case of the Stokes equations. Since the Stokes equations are the "limit" as  $\nu \rightarrow 1/2$  of the equations of elasticity this also intuitively explains the lack of uniformity in  $\nu$  of the accuracy of the approximation (5.2) to the equations of elasticity. (Note, however, the difference in our boundary conditions.)

The second equation in (5.1) requires that  $\underline{U}_h$  lies in the subspace

$$(5.4) \quad Z_h = \{\underline{v} \in \overset{\circ}{V}_h : b(\underline{v}, q) = 0 \quad \forall q \in \Pi_h\}$$

and part of (5.1) may thus be restated

$$(5.5) \quad \begin{cases} \text{Find } \underline{U}_h \in Z_h \text{ such that} \\ a(\underline{U}_h, \underline{v}) = (\underline{F}, \underline{v}) \text{ for all } \underline{v} \in Z_h. \end{cases}$$

The reason for the deficiency of the choice of spaces  $\overset{\circ}{V}_h$  and  $\Pi_h$ , given by

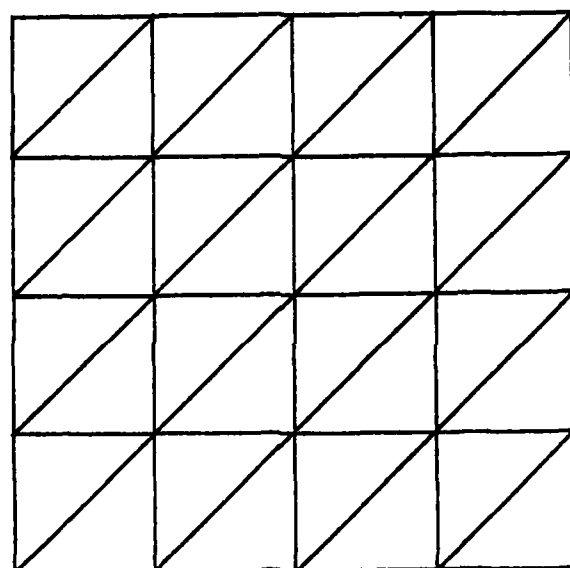


Figure 1

Uniform triangulation of size  $h$  ,  $\sum_h^1$  .

(5.3) is that the corresponding  $Z_h$  on the mesh  $\sum_h^1$  consists of the zero element only. This can be seen as follows: we first observe that

$$Z_h = \nabla \times \{C^1 \text{ piecewise quadratics all of whose first derivatives vanish on } \partial\Omega\},$$

with the curl operator  $\nabla \times$  given by

$$\nabla \times \psi = \left( \frac{\partial}{\partial x_2} \psi, -\frac{\partial}{\partial x_1} \psi \right).$$

A  $C^1$  piecewise quadratic whose first derivatives vanish on  $\partial\Omega$  can be extended by constants onto  $\mathbb{R}^2 \setminus \Omega$ . By subtracting the constant which is attained in the unbounded component of  $\mathbb{R}^2 \setminus \Omega$  we thus obtain a  $C^1$  piecewise quadratic with compact support. On the mesh shown in Fig. 1 there is only one piecewise quadratic with compact support (cf. Morgan & Scott [26], Chui & Wang [7]) namely the constant 0! Consequently it follows that

$$\begin{aligned} Z_h &= \nabla \times \{\text{constant function}\} \\ &= \{0\}. \end{aligned}$$

The formulation (5.5) mimics the following equivalent version of the Stokes problem

$$(5.6) \quad \begin{cases} \text{Find } \underline{U} \in Z \text{ such that} \\ a(\underline{U}, \underline{v}) = (\underline{F}, \underline{v}) \text{ for all } \underline{v} \in Z \end{cases}$$

where  $Z$  denotes the subspace

$$\begin{aligned} Z &= \{\underline{v} \in [H^1(\Omega)]^2 : b(\underline{v}, q) = 0 \quad \forall q \in \mathring{L}_2(\Omega)\} \\ &= \{\underline{v} \in [H^1(\Omega)]^2 : \nabla \cdot \underline{v} = 0\}. \end{aligned}$$



Note that (5.6) represents the well known Hodge decomposition (cf. Temam [35]).

The space  $Z_h$ , as defined in (5.4), is of course not necessarily contained in

$Z$ , but in many interesting cases this inclusion holds, i.e.  $Z_h = \hat{V}_h \cap Z$ .

Since  $a(\cdot, \cdot)$  is a positive definite symmetric form that coerces the  $H^1$ -norm (on  $\hat{H}^1(\Omega)$ ) it follows in this case that

$$\|\underline{U} - \underline{U}_h\|_{H^1(\Omega)} \leq C \inf_{\underline{z} \in Z_h} \|\underline{U} - \underline{z}\|_{H^1(\Omega)}.$$

We would thus obtain a quasioptimal velocity approximation,  $\underline{U}_h$ , if the following condition were to hold:

For any  $\underline{U} \in Z$ ,

$$(5.7) \quad \inf_{\underline{z} \in Z_h} \|\underline{U} - \underline{z}\|_{H^1(\Omega)} \leq C \inf_{\underline{v} \in \hat{V}_h} \|\underline{U} - \underline{v}\|_{H^1(\Omega)},$$

with  $C$  independent of  $\underline{U}$  and  $h$ .

#### Definition I

A family of closed (not necessarily finite dimensional) subspaces  $W_h \subseteq [H^1(\Omega)]^2$  is called divergence-stable if

- i) the spaces  $\nabla \cdot W_h$  are closed in  $L_2(\Omega)$ ,
- ii)  $\exists c > 0$ , independent of  $h$ , such that

$$\sup_{\underline{w} \in W_h \setminus \{0\}} \frac{b(\underline{w}, q)}{\|\underline{w}\|_{H^1(\Omega)}} \geq c \|q\|_{L_2(\Omega)}$$

for all  $q \in \nabla \cdot W_h$ .

Remark 5.1

The definition of divergence-stability as given above is equivalent to the requirement that there exists a uniformly bounded maximal right inverse for the divergence operator on the spaces  $W_h$ , i.e. there exists a family of linear operators

$$L_h : \nabla \cdot W_h \rightarrow W_h$$

such that

$$i) \quad \nabla \cdot (L_h q) = q, \quad \forall q \in \nabla \cdot W_h \quad \text{and}$$

$$ii) \quad \|L_h q\|_{1,\Omega} \leq C \|q\|_{0,\Omega},$$

with a constant  $C$  that is independent of  $h$  and  $q$ .

The condition (5.7) and the concept of divergence-stability are intimately related as shown by the following result.

Proposition 5.1

The spaces  $Z_h = \mathring{V}_h \cap Z$  and  $\mathring{V}_h$  satisfy (5.7) for any  $\underline{U} \in Z$ , with a constant  $C$  that is independent of  $h$  and  $\underline{U}$ , if and only if  $\mathring{V}_h$  is divergence-stable.

Proof:

Assume that  $Z_h = \mathring{V}_h \cap Z$  and  $\mathring{V}_h$  satisfy (5.7). For any  $q \in \nabla \cdot \mathring{V}_h$  there exists

$$\underline{v} \in [H^1(\Omega)]^2,$$

such that

$$\nabla \cdot \underline{v} = q \quad \text{and} \quad \|\underline{v}\|_{H^1(\Omega)} \leq C \|q\|_{L_2(\Omega)}$$

(cf. Temam [35]).

Since  $q \in \nabla \cdot \mathring{V}_h$  there also exists  $\underline{v}_h \in \mathring{V}_h$  such that

$$\nabla \cdot \underline{v}_h = q.$$

Hence

$$\underline{v} - \underline{v}_h \in Z,$$

and therefore there exists  $\underline{z}_h \in Z_h$  with

$$\begin{aligned} (5.8) \quad \|\underline{v} - \underline{v}_h - \underline{z}_h\|_{H^1(\Omega)} &\leq C \inf_{\underline{w} \in \mathring{V}_h} \|\underline{v} - \underline{v}_h - \underline{w}\|_{H^1(\Omega)} \\ &\leq C \|\underline{v}\|_{H^1(\Omega)}. \end{aligned}$$

The estimate (5.8) immediately leads to

$$\begin{aligned} (5.9) \quad \|\underline{v}_h + \underline{z}_h\|_{H^1(\Omega)} &\leq C \|\underline{v}\|_{H^1(\Omega)} \\ &\leq C \|q\|_{L_2(\Omega)}. \end{aligned}$$

Since also

$$(5.10) \quad \nabla \cdot (\underline{v}_h + \underline{z}_h) = q$$

the field  $\underline{w}_h = -(\underline{v}_h + \underline{z}_h) \in \mathring{V}_h$  satisfies

$$(5.11) \quad \frac{b(\underline{w}_h, q)}{\|\underline{w}_h\|_{H^1(\Omega)}} = \frac{\|q\|_{L_2(\Omega)}^2}{\|\underline{w}_h\|_{H^1(\Omega)}} \geq c \|q\|_{L_2(\Omega)}.$$

As a consequence of (5.9)-(5.11) we conclude that  $\mathring{V}_h$  is divergence-stable.

The proof in the opposite direction is a simple consequence of the results of Babuska [2] and Brezzi [6]; for completeness we include the details of a proof here. For each  $\underline{v} \in \mathring{V}_h$ , let  $\underline{z}_h \in Z_h = \mathring{V}_h \cap Z$  be its orthogonal projection with respect to the inner-product  $a(\cdot, \cdot)$ , and define  $q_h \in \nabla \cdot \mathring{V}_h$  via

$$b(\underline{w}, q_h) = a(\underline{v} - \underline{z}_h, \underline{w}) \text{ for all } \underline{w} \in \mathring{V}_h.$$

Since  $\mathring{V}_h$  is divergence-stable,  $q_h$  is well defined and satisfies

$$\|q_h\|_{L_2(\Omega)} \leq C \|\underline{v} - \underline{z}_h\|_{H^1(\Omega)}.$$

Now by definition of  $\underline{z}_h$  and  $q_h$ ,

$$\begin{aligned} c \|\underline{v} - \underline{z}_h\|_{H^1(\Omega)}^2 &\leq a(\underline{v} - \underline{z}_h, \underline{v} - \underline{z}_h) \\ &= a(\underline{v} - \underline{z}_h, \underline{v}) \\ &= b(\underline{v}, q_h) \\ &= b(\underline{v} - \underline{u}, q_h) \\ &\leq C \|\underline{v} - \underline{u}\|_{H^1(\Omega)} \|q_h\|_{L_2(\Omega)} \\ &\leq C \|\underline{v} - \underline{u}\|_{H^1(\Omega)} \|\underline{v} - \underline{z}_h\|_{H^1(\Omega)} \end{aligned}$$

for any  $\underline{u} \in Z$ . Thus we have, for any  $\underline{u} \in Z$ ,

$$\|\underline{v} - \underline{z}_h\|_{H^1(\Omega)} \leq C \|\underline{v} - \underline{u}\|_{H^1(\Omega)}.$$

By the triangle inequality, this means that, for  $\underline{u} \in Z$  and  $\underline{v} \in \mathring{V}_h$ ,

$$\begin{aligned}
 (5.12) \quad \|u - z_h\|_{H^1(\Omega)} &\leq \|u - v\|_{H^1(\Omega)} + \|v - z_h\|_{H^1(\Omega)} \\
 &\leq C \|u - v\|_{H^1(\Omega)}
 \end{aligned}$$

provided  $z_h$  is the projection (relative to the inner product  $a$ ) of  $v$  onto  $Z_h = \hat{V}_h \cap Z$ . The estimate (5.12) clearly implies (5.7). (Note that at no point in this proof did we use that  $\hat{V}_h$  is finite dimensional).

Remark 5.2

Proposition 5.1 does not assert that divergence-stability is always necessary in order to get optimal-rate velocity approximations. Indeed optimal-rate velocity approximations are in certain cases achieved with piecewise polynomial spaces  $\hat{V}_h$  that are not divergence-stable (cf. Remark 6.1).

We now suppose that  $u_h \in Z_h$  is known through the solution of (5.5), and we consider determining  $P_h \in \Pi_h$  from the first equation in (5.1)

$$(5.13) \quad b(v, P_h) = (F, v) - a(u_h, v) \quad \text{for all } v \in \hat{V}_h.$$

In order for  $P_h$  to be unique it is necessary and sufficient that there exists  $c > 0$  such that

$$(5.14) \quad \sup_{v \in \hat{V}_h \setminus \{0\}} \frac{b(v, q)}{\|v\|_{H^1(\Omega)}} \geq c \|q\|_{L_2(\Omega)}$$

for all  $q \in \Pi_h$ . If we furthermore want  $\|P_h\|_{L_2(\Omega)}$  uniformly bounded by  $C(\|P\|_{L_2(\Omega)} + \|u\|_{H^1(\Omega)})$  independently of  $h$  then it suffices to require that the constant  $c$  in (5.14) is independent of  $h$ .

We note that although it is mathematically simple to describe it is in practice not always obvious how to use the equations (5.5) and (5.13) to solve for  $u_h$  and  $P_h$  (cf. Glowinski [14]).

Definition II

Let  $W_h$  and  $Q_h$  be two families of closed subspaces of  $[H^1(\Omega)]^2$  and  $L_2(\Omega)$  respectively. The family  $W_h$  is said to be divergence-stable relative to  $Q_h$  if  $\exists c > 0$ , independent of  $h$ , such that

$$\sup_{\underline{w} \in W_h \setminus \{0\}} \frac{b(\underline{w}, q)}{\|\underline{w}\|_{H^1(\Omega)}} \leq c \|q\|_{L_2(\Omega)}$$

for all  $q \in Q_h$ .

Remark 5.3

By comparison of the definitions I and II we note that a family of closed subspaces  $W_h \subseteq [H^1(\Omega)]^2$  is divergence-stable if and only if it is divergence-stable relative to the family  $\nabla \cdot W_h$ .

Using the concepts of divergence-stability introduced in definitions I and II the question of stability of finite element approximations to the Stokes equations (2.1) and the equations of elasticity (3.1) is well understood.

Proposition 5.2

Suppose that the family of spaces  $\mathring{V}_h \subseteq [H^1(\Omega)]^2$  is divergence-stable relative to the family  $\Pi_h \subseteq \mathring{L}_2(\Omega)$ . Let  $(\underline{U}, P) \in [\mathring{H}^1]^2 \times \mathring{L}_2$  and  $(\underline{U}_h, P_h) \in \mathring{V}_h \times \Pi_h$  denote the solutions to (4.4) and (5.1) respectively; then

$$\begin{aligned} & \|\underline{U} - \underline{U}_h\|_{H^1(\Omega)} + \|P - P_h\|_{L_2(\Omega)} \\ & \leq C \left( \inf_{\underline{v} \in \mathring{V}_h} \|\underline{U} - \underline{v}\|_{H^1(\Omega)} + \inf_{q \in \Pi_h} \|P - q\|_{L_2(\Omega)} \right) \end{aligned}$$

and  $C$  is independent of  $h$  (and  $\underline{F}$ ).

Proposition 5.2 follows directly from the results of Babuška [2] or Brezzi [6]. It reduces the question of convergence to one of approximation theory and this inherently leads to contradictory requirements: on one

hand the spaces  $\Pi_h$  should have good approximation properties - on the other hand they should not be too big, since that would jeopardize stability. One very natural choice of  $\Pi_h$  seems to be  $\Pi_h = \nabla \cdot \overset{\circ}{V}_h$ .

Proposition 5.3

Suppose that the family of spaces  $\overset{\circ}{V}_h \subseteq [H^1(\Omega)]^2$  is divergence-stable, and choose  $\Pi_h = \nabla \cdot \overset{\circ}{V}_h$ . Let  $(\underline{U}, P) \in [H^1]^2 \times L_2$  and  $(\underline{U}_h, P_h) \in \overset{\circ}{V}_h \times \Pi_h$  be the solutions to (4.4) and (5.1) respectively; then

$$i) \quad \|\underline{U} - \underline{U}_h\|_{H^1(\Omega)} \leq C \inf_{\underline{v} \in \overset{\circ}{V}_h} \|\underline{U} - \underline{v}\|_{H^1(\Omega)} \quad \text{and}$$

$$ii) \quad \|P - P_h\|_{L_2(\Omega)} \leq C \left( \inf_{\underline{v} \in \overset{\circ}{V}_h} \|\underline{U} - \underline{v}\|_{H^1(\Omega)} + \inf_{q \in \Pi_h} \|P - q\|_{L_2(\Omega)} \right)$$

and  $C$  is independent of  $h$  (and  $\underline{F}$ ).

Proof

This follows immediately from Prop. 5.1, Prop. 5.2 and the analysis leading to the definition I.

In the following we shall denote by  $\underline{U}^v \in [H^1]^2$ ,  $\underline{U}_h^v \in \overset{\circ}{V}_h$  the solutions to (4.5) and (5.2), thus emphasizing the dependence on  $0 < v < 1/2$  of the solution to the problem of elasticity.

Proposition 5.4

Suppose that the family of spaces  $\overset{\circ}{V}_h \subseteq [H^1(\Omega)]^2$  is divergence-stable. Then

$$\begin{aligned} & \|\underline{U}^v - \underline{U}_h^v\|_{H^1(\Omega)} + \frac{1}{1-2v} \|\nabla \cdot (\underline{U}^v - \underline{U}_h^v)\|_{L_2(\Omega)} \\ & \leq C \left( \inf_{\underline{v} \in \overset{\circ}{V}_h} \|\underline{U}^v - \underline{v}\|_{H^1(\Omega)} + \inf_{q \in \nabla \cdot \overset{\circ}{V}_h} \left\| \frac{1}{1-2v} \nabla \cdot \underline{U}^v - q \right\|_{L_2(\Omega)} \right) \end{aligned}$$

with a constant  $C$  that is independent of  $h$  and  $v$  (and  $\underline{F}$  and  $\underline{g}$ ).

For a fixed  $0 < \nu < 1/2$  this estimate with a constant  $C_\nu$ , possibly depending on  $\nu$ , follows directly from the results of Babuška [2]. The fact that  $C$  may be chosen independently of  $\nu$ , provided the family  $V_h$  is divergence-stable, is e.g. proven in Vogelius [36].

Several ways to remedy the deficiency of the spaces in (5.3) have been suggested and analyzed in the literature. A non-conforming, piecewise linear, triangular element was proposed by Crouzeix and Raviart [10] which, together with the  $\Pi_h$  in (5.3) yields optimal approximation (second order). For computational experience with this element, see Pritchard, Renardy & Scott [31] and references therein. Increasing  $Z_h$  can also be achieved by choosing a special triangulation, as shown in Figure 2, although this is by no means obvious. The mesh consists of squares with diagonals drawn in; the intersections of the diagonals are special examples of what we shall denote "singular vertices" (the exact definition of a singular vertex will be given in the next section).

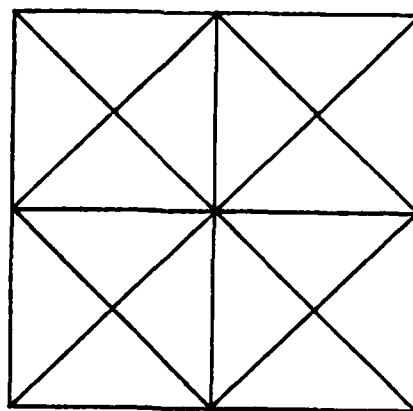


Figure 2

Special mesh  $\sum_h^2$ , with "singular vertices"



It was discovered by Powell [30] that the space of  $C^1$  piecewise quadratics on the mesh in Figure 2 has good approximation properties - there is roughly one degree of freedom per singular vertex. One must reduce  $\Pi_h$  in (5.3) appropriately at singular vertices, cf. later in this paper. For computational experience with, and analysis of, this method see Fix, Gunzburger & Nicolaides [13], Malkus & Olsen [22], and Mercier [24] and references therein.

From the definition II it follows that a possible method to achieve divergence-stability of  $W_h$  relative to  $Q_h$  (if it is not already there) is to increase  $W_h$ , or decrease  $Q_h$  (or both). The well-known Hood-Taylor [18] element is obtained from continuous piecewise quadratic velocities, piecewise linear pressures exactly this way by requiring the pressures also to be continuous. The element analyzed by Arnold, Brezzi & Fortin [1] has a similar flavor: for velocities the continuous piecewise linears are enriched by cubic bubble functions (the pressures however are taken to be continuous piecewise linear).

In the context of the equations of elasticity it has been proposed to use "reduced integration" on the term involving  $b$ , while maintaining simple choices for both the meshes and  $V_h$ . We refer to Zienkiewicz [38], Malkus & Hughes [21] and references therein for computational experience with reduced integration methods, and an account of their relation to mixed methods.

An obvious question at this point seems to be:

"are the problems that we encountered with conforming piecewise linear elements also present for conforming piecewise polynomials of degree  $p + 1$ ,  $p \geq 1$ ?"

This question is the focus of the next two sections.

## 6. Divergence-stability of high-order conforming spaces

Let us consider the spaces

$$V_h = [P_h^{[p+1],0}]^2 = [C^0 \text{ piecewise polynomials of degree } \leq p+1]^2 \quad (6.1)$$

$$\hat{V}_h = [\hat{P}_h^{[p+1],0}]^2 = \{\underline{v} \in [P_h^{[p+1],0}]^2 : \underline{v} = 0 \text{ on } \partial\Omega\}$$

on an arbitrary family of triangulations  $\sum_h$  of  $\Omega$ . We have already seen that for  $p = 0$  these spaces are not in general divergence-stable. Results due to de Boor, Höllig and Jia [11], [19] show that these spaces are not in general divergence-stable for  $p = 1$  or  $2$ ; this lack of divergence-stability is well-documented by computational experience (cf. section 7 for more details). We now turn our attention to the case  $p \geq 3$ . In order to state rigorously a result concerning the divergence-stability of the choice (6.1) for  $p \geq 3$  we need some notation. A singular internal vertex [25] is one where precisely four triangles meet through the intersection of two straight lines, as shown in Figure 3. A singular boundary vertex is a vertex on  $\partial\Omega$  where  $1 \leq k \leq 4$  triangles meet through the intersection of two straight lines. There are

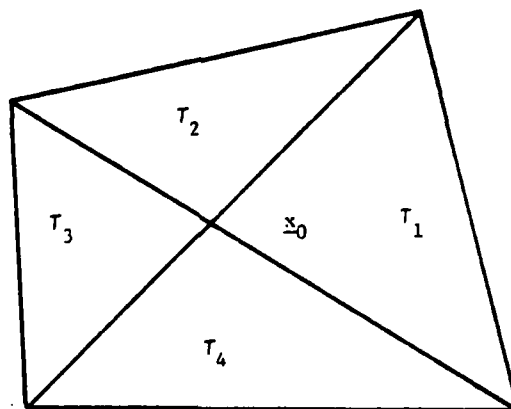


Figure 3

Singular internal vertex.

four such possibilities as shown in Figure 4.

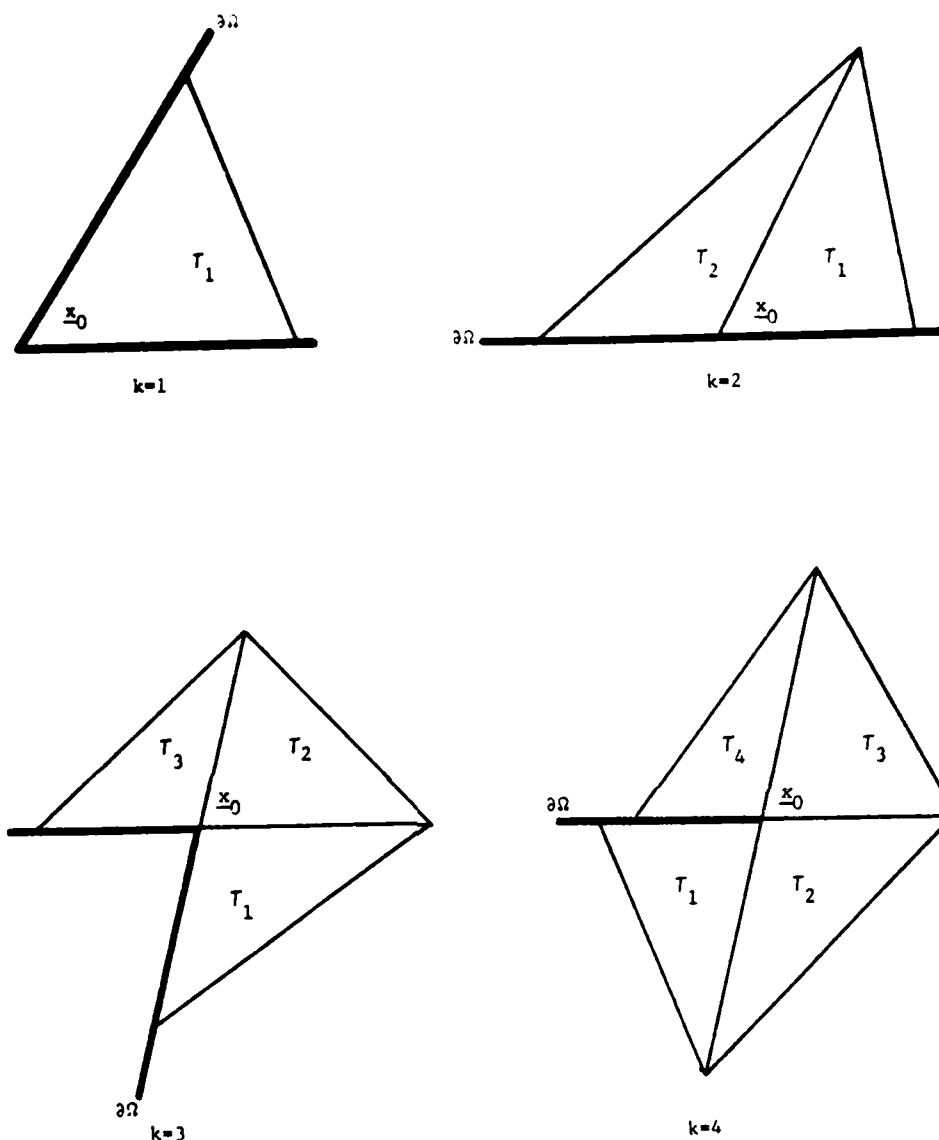


Figure 4

Singular boundary vertices.

**Theorem 6.1** [32]

Suppose that  $\sum_h$  is quasi-uniform and that no nonsingular vertices degenerate towards singular as  $h$  tends to zero. Then for any fixed  $p \geq 3$  the spaces (6.1) are divergence stable (in the sense of Definition I).

Furthermore the constant  $c$  entering into Definition I is bounded from below by

$$c'p^{-k}, \quad c' > 0$$

with  $c'$  and  $k$  independent of  $h$  and  $p$ .

Remark 6.1

By quasi-uniform we mean that each triangle  $T \in \sum_h$  contains a ball of radius  $\rho h$ , where  $\rho > 0$  is independent of  $h$ . Whether this restriction is essential to guarantee divergence-stability of the spaces (6.1) for  $p \geq 3$ , we do not know. It is easy to see that the spaces (6.1) cannot be divergence-stable if a nonsingular vertex tends towards becoming singular (as seen later the dimension of the space  $\nabla \cdot V_h$  or  $\nabla \cdot \overset{\circ}{V}_h$  decreases by one when a vertex becomes singular). This does not imply that we will observe sub-optimal convergence-rates when discretizing the Stokes problem or the equations of elasticity,  $\nu \sim 1/2$ , using a mesh with nearly-singular (nonsingular) vertices, however, the ratio between the error in the finite element approximation of the velocities (or displacements) and the error of the best  $(H^1-)$  approximation may be arbitrarily large depending on the data. As stated in Theorem 6.1 the lower bound in the estimate of divergence-stability may approach zero algebraically in  $p^{-1}$  as  $p \uparrow \infty$ , we do not know whether this is indeed the case or whether the lower bound is uniform in  $p$  also. Note that divergence stability of  $V_h$  requires only that nonsingular internal vertices do not degenerate to singularity as  $h \rightarrow 0$ .

For the proof of the theorem, as well as more details concerning the results, we refer to Scott & Vogelius [32] and Vogelius [37]. In case of the equations of elasticity (4.5) Theorem 6.1 in combination with Proposition 5.4 leads to the fact that the spaces (6.1) give quasi-optimal finite element approximations to  $(\underline{U}^v, \frac{1}{1-2\nu} \nabla \cdot \underline{U}^v)$  for fixed  $p \geq 3$ . The question of convergence properties thus becomes one of approximation theory, and in addition to understanding the approximation properties of  $V_h$  it is

important to understand the character of the spaces  $\nabla \cdot V_h$ . The fact that the lower bound in the stability estimate approaches zero at most algebraically in  $p^{-1}$  as  $p \uparrow \infty$  was used in Vogelius [36] to prove "almost" optimal convergence rates for the so-called p-version of the finite element method. Using an interpolation trick it was shown (on a smooth domain, with curved elements at the boundary) that the p-version converges at "almost" optimal rates in the energy norm, uniformly with respect to Poisson's ratio. "Almost" optimal here means that it converges at any rate strictly less than optimal. A slight variation of the present result (permitting for curved elements at the boundary) could likewise be used to verify a similar result for combinations of the h- and p-versions (i.e. when simultaneously changing mesh size and degree of the polynomials).

For the case of the Stokes problem it is clear from both Proposition 5.2 and Proposition 5.3 that in addition to divergence-stability and good approximation properties of the spaces  $\overset{\circ}{V}_h$  it is important to have good approximation properties of the corresponding pressure-spaces  $\Pi_h$ . Since Theorem 6.1 guarantees divergence-stability relative to any subspace of  $\overset{\circ}{\nabla \cdot V}_h$  this naturally leads to the question of finding a characterization of the spaces  $\overset{\circ}{\nabla \cdot V}_h$ . Part of this characterization (its necessity) is implicit in the work of Nagtegaal, Parks & Rice [27] and that of Mercier [23].

Let  $\underline{x}_0$  be an internal vertex where four triangles meet with the common edges lying on either the  $x_1$ -axis or the  $x_2$ -axis, as shown in Figure 5.

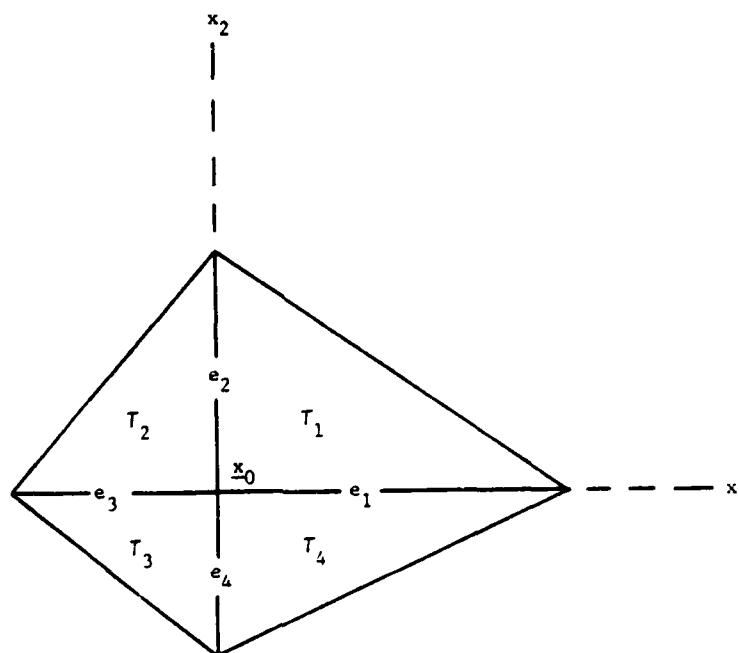


Figure 5

Let  $\underline{v} \in V_h$  and set  $\underline{v}^i = \underline{v}|_{T_i}$   $i = 1, \dots, 4$ . Since  $\underline{v}$  is continuous  $v_1^1 - v_1^4$  vanishes identically on the edge  $e_1$ , consequently

$$(6.2a) \quad \frac{\partial}{\partial x_1} v_1^1 = \frac{\partial}{\partial x_1} v_1^4 \quad \text{on } e_1.$$

Similarly we get

$$(6.2b) \quad \frac{\partial}{\partial x_1} v_1^3 = \frac{\partial}{\partial x_1} v_1^2 \quad \text{on } e_3,$$

$$(6.2c) \quad \frac{\partial}{\partial x_2} v_2^1 = \frac{\partial}{\partial x_2} v_2^2 \quad \text{on } e_2, \quad \text{and}$$

$$(6.2d) \quad \frac{\partial}{\partial x_2} v_2^3 = \frac{\partial}{\partial x_2} v_2^4 \quad \text{on } e_4.$$

The vertex  $\underline{x}_0$  is common to all the edges  $e_1$  through  $e_4$  and by summation of (6.2a-d) at  $\underline{x}_0$  we get

$$(6.3) \quad \nabla \cdot \underline{v}^1(\underline{x}_0) + \nabla \cdot \underline{v}^3(\underline{x}_0) = \nabla \cdot \underline{v}^2(\underline{x}_0) + \nabla \cdot \underline{v}^4(\underline{x}_0),$$

which, by introduction of  $\phi = \nabla \cdot \underline{v}$ , may be restated

$$(6.4) \quad \sum_{i=1}^4 (-1)^i \phi|_{T_i}(\underline{x}_0) = 0.$$

The chain rule of differentiation gives that

$$(6.5) \quad \nabla \cdot (T^{-1} \underline{v} \circ T)(\underline{x}) = (\nabla \cdot \underline{v})(T\underline{x})$$

for any invertible affine map  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ . If  $\underline{v} \in V_h$  then so is  $T^{-1} \underline{v}(T\underline{x})$  (on the corresponding triangulation) and it thus follows immediately from (6.5) that (6.3) (or (6.4)) must be satisfied at all internal vertices which can be obtained by an affine transformation of one as in Figure 5. These vertices are exactly the singular internal vertices.

This simple calculation thus shows that for any  $p \geq 0$

$$(6.6) \quad \nabla \cdot V_h \subseteq p_h^{[p], -1} = \{(\text{discontinuous}) \text{ piecewise} \\ \text{polynomials, } \phi, \text{ of degree } \leq p \\ \text{that satisfy (6.4) at singular} \\ \text{internal vertices}\}.$$

For the spaces  $\overset{\circ}{V}_h$ , with homogeneous Dirichlet boundary conditions, we find two additional sets of constraints, namely with  $\phi = \nabla \cdot \underline{v}$ :

$$(6.7) \quad \sum_{i=1}^k (-1)^i \phi|_{T_i}(\underline{x}_0) = 0$$

at any singular boundary vertex  $\underline{x}_0$ ,  
 where  $1 \leq k \leq 4$  triangles meet as in  
 Figure 4, and

$$(6.8) \quad \int_{\Omega} \phi \, d\underline{x} = 0$$

If we define

$\tilde{p}_h^{[p],-1} = \{(\text{discontinuous}) \text{ piecewise polynomials,}$   
 $\phi$ , of degree  $\leq p$ , that satisfy (6.4) at singular  
 internal vertices, (6.7) at singular boundary  
 vertices and furthermore satisfy (6.8)}\}

then this shows that for any  $p \geq 0$

$$(6.9) \quad \nabla \cdot \overset{\circ}{V}_h \subseteq \tilde{p}_h^{[p],-1}.$$

It was shown in Vogelius [37] that (6.6) is indeed an equality provided  $p \geq 3$ .  
 By a similar combinatorial argument it is proven in Scott & Vogelius [32]  
 that (6.9) is also an equality for any  $p \geq 3$  (remember,  $\Omega$  is connected). This  
 characterizes the spaces  $\nabla \cdot V_h$  and  $\nabla \cdot \overset{\circ}{V}_h$  on an arbitrary triangulation for any  $p \geq 3$ .

Remark 6.2

From the characterization of  $\nabla \cdot \overset{\circ}{V}_h$  given above it follows that  
 the space

$$(6.10) \quad \{\text{continuous piecewise polynomials of degree } \leq p \\ \text{that satisfy (6.8)}\}$$



is a subspace of  $\nabla \cdot \overset{\circ}{V}_h$  provided  $p \geq 3$ . Selecting the pressure space  $\Pi_h$  to be as in (6.10),  $p \geq 3$ , generalizes the methods studied by Hood & Taylor [18] for  $p = 1$  and  $2$ . Theorem 6.1 shows that these generalized Hood-Taylor elements,  $p \geq 3$ , lead to quasi-optimal approximation. Bercovier & Pironneau [3] proved the same to be the case for the Hood-Taylor element for  $p = 1$ . It would be natural to conjecture that the case  $p = 2$  also leads to quasi-optimal approximations.

Remark 6.3

If one uses the approach (5.5) to compute the finite element approximation to the velocity  $\underline{U}$  in the Stokes problem then it is important to have a local basis for  $Z_h$ . Assuming that  $\Pi_h$  has been selected so that  $Z_h \subseteq Z$  it then follows that, for all  $p \geq 0$ ,

$$(6.11) \quad Z_h = \{\underline{v} \in \overset{\circ}{V}_h : \nabla \cdot \underline{v} = 0\} = \nabla \times \overset{\circ}{P}_h^{[p+2],1},$$

where

$$\overset{\circ}{P}_h^{[p+2],1} = \{C^1 \text{ piecewise polynomials of degree } \leq p+2 \text{ all of whose first derivatives vanish at the boundary}\}.$$

Note that elements of  $\overset{\circ}{P}_h^{[p+2],1}$  are constant on each connected component of  $\partial\Omega$ , however the constants need not be the same; if  $\partial\Omega$  is connected then we shall always pick the constant to be zero for functions in  $\overset{\circ}{P}_h^{[p+2],1}$ .

A basis for  $\overset{\circ}{P}_h^{[p+2],1}$ ,  $p \geq 3$ , (with no boundary conditions) was constructed in Morgan & Scott [25]; the corresponding basis for  $\overset{\circ}{P}_h^{[p+2],1}$ ,  $p \geq 3$ , is not altogether obvious, but is described in Scott & Vogelius [32]. The finite element method corresponding to the spaces  $\overset{\circ}{V}_h$  and  $\Pi_h = \nabla \cdot \overset{\circ}{V}_h$  in the case  $p = 3$  is currently being tested numerically for the full (nonlinear, time-dependent) Navier-Stokes equations.

## 7. Results concerning lower-degree spaces

There are really three aspects of the finite element approximation to the Stokes-equations or the equations of elasticity addressed in the previous sections.

- a) Bounds for a maximal right inverse for the divergence operator on  $V_h$  and  $\overset{\circ}{V}_h$ , or estimates for the divergence-stability of these spaces relative to families of pressure spaces  $\Pi_h$ .
- b) Characterization of the spaces  $\nabla \cdot V_h$  or  $\nabla \cdot \overset{\circ}{V}_h$ .
- c) Determination of the approximation properties of the spaces  $\overset{\circ}{V}_h \cap \{\underline{v}: \nabla \cdot \underline{v} = 0\}$  (or  $V_h \cap \{\underline{v}: \nabla \cdot \underline{v} = 0\}$ ).

For the equations of elasticity (4.5) we are as indicated by Proposition 5.4 mainly concerned with a) and b) whereas the aspect c) is also of importance when solving the Stokes-problem (4.4). As pointed out in sections 5 and 6 (particularly in Remark 5.2 and Remark 6.1) c) may be of independent interest even though a) and b) do not have satisfactory answers. We shall now discuss the aspects a)-c) for the spaces

$$(7.1) \quad V_h = [P_h^{[p+1]}, 0]^2 \quad \text{and} \quad \overset{\circ}{V}_h = [\overset{\circ}{P}_h^{[p+1]}, 0]^2, \quad p = 0, 1, 2.$$

We know of very few results that are valid on a quite arbitrary triangulation for these low degree spaces; as a consequence we shall restrict our attention to the triangulations, a local picture of which are shown in Figure 1 and Figure 2 (we denote these  $\sum_h^1$  and  $\sum_h^2$  respectively). On these triangulations the dimension formula conjectured by Strang [33]

$$(7.2) \quad \dim(P_h^{[p+2], 1}) = \frac{1}{2}(p+3)(p+4)T - (2p+5)E_0 + 3V_0 + \sigma_0$$

is known to hold also for  $p = 0, 1$  and  $2$  (cf. Morgan & Scott [26]). Here  $T$  denotes the number of triangles,  $E_0$  is number of internal edges,  $V_0$  is the number of internal vertices and  $\sigma_0$  is the number of singular internal vertices. On a simply connected domain the null space of the divergence operator acting on  $V_h = [P_h^{[p+1],0}]^2$  is the curl of  $P_h^{[p+2],1}$ ; it thus follows as in Vogelius [37] (or Scott & Vogelius [32]) that whenever (7.2) holds  $\nabla \cdot V_h$  must have the same dimension  $\frac{1}{2}(p+2)(p+1)T - \sigma_0$  as the space  $P_h^{[p],-1}$  (cf. (6.6)). Since  $\nabla \cdot V_h \subseteq P_h^{[p],-1}$  we conclude that

$$(7.3) \quad \nabla \cdot V_h = P_h^{[p],-1}$$

whenever (7.2) holds for a given triangulation, and in particular for  $p = 0, 1$  and  $2$  on the triangulations  $\sum_h^1$  and  $\sum_h^2$ . (For  $p = 2$  we could indeed have derived (7.2) for much more general triangulations, cf. Morgan & Scott [26].) By a hole-filling procedure we can extend our argument to verify (7.3) even though the domain is not simply connected. The characterization (7.3) for the case  $p = 0$  and the triangulation  $\sum_h^2$  was also noted by Fix, Gunzburger & Nicolaides (cf. [13]).

In contrast the relation between the spaces  $\nabla \cdot \mathring{V}_h$  and  $\mathring{P}_h^{[p],-1}$  is not nearly as simple for  $p = 0, 1$  and  $2$  as the characterization given in the previous section, for  $p \geq 3$ , might lead one to believe. Let us start by considering the piecewise linear case,  $p = 0$ . If  $\sigma$  denotes the total number of singular vertices, including singular boundary vertices, then we have for the triangulations  $\sum_h^1$  and  $\sum_h^2$

$$\dim \mathring{V}_h = 2V_0 \quad \text{and} \quad \dim \mathring{P}_h^{[0],-1} = T - \sigma - 1$$

(excluding the trivial case of a rectangle divided into two triangles by the

diagonal, when both  $\overset{\circ}{V}_h$  and  $\tilde{p}_h^{[0],-1}$  consist of 0 only). Using the relations

$$E - E_0 = V - V_0, \quad E + E_0 = 3T$$

and Euler's formula

$$T + V - E = 1$$

(assuming  $\Omega$  is simply connected), we get

$$(7.4) \quad \dim \tilde{p}_h^{[0],-1} - \dim \overset{\circ}{V}_h = (E - E_0) - \sigma - 3;$$

$T$ ,  $E_0$ ,  $V_0$  are as before and  $E$  denotes the total number of edges,  $V$  the total number of vertices. From (7.4) we immediately conclude that if  $\sigma < E - E_0 - 3$  then  $\dim \overset{\circ}{V}_h < \dim \tilde{p}_h^{[0],-1}$  and consequently  $\nabla \cdot \overset{\circ}{V}_h$  is a proper subspace of  $\tilde{p}_h^{[0],-1}$  (see Nagtegaal, Parks & Rice [27] and Malkus & Hughes [21] for a similar constraint-counting method). If  $\Omega$  is a rectangle and  $\sum_h^1$  is used for a triangulation then there are exactly 2 singular vertices (one in the upper-left and one in the lower-right corner of  $\Omega$ ); thus  $\nabla \cdot \overset{\circ}{V}_h$  is a proper subspace of  $\tilde{p}_h^{[0],-1}$  except for the trivial case that  $\Omega$  is divided into only two triangles. Since  $\overset{\circ}{V}_h \cap Z = \{\underline{v} \in \overset{\circ}{V}_h : \nabla \cdot \underline{v} = 0\} = \{0\}$  on the mesh  $\sum_h^1$  it follows that  $\nabla \cdot$  is injective on  $\overset{\circ}{V}_h$  and hence we get from (7.4) that

$$\dim \tilde{p}_h^{[0],-1} - \dim \nabla \cdot \overset{\circ}{V}_h = (E - E_0) - 5$$

on a rectangle triangulated by the mesh  $\sum_h^1$  (excluding the trivial case of only 2 triangles).

In the case that  $\Omega$  is a rectangle divided into triangles by the mesh  $\sum_h^2$  then the dimension of  $\overset{\circ}{p}_h^{[2],1} = \overset{\circ}{p}_h^{[2],1} \cap \{\psi : \psi = \frac{\partial \psi}{\partial n} = 0 \text{ on } \partial\Omega\}$

has been calculated by Chui, Schumaker & Wang [8] to be

$$\dim \tilde{p}_h^{[p],1} = \sigma - (E - E_0) + 4 ,$$

provided  $\Omega$  has at least two boundary edges on each side. As a consequence of this and (6.11) it follows that

$$\dim \tilde{V}_h^\circ = \dim \nabla \cdot \tilde{V}_h^\circ + \sigma - (E - E_0) + 4 ,$$

and thus, using (7.4), we find that

$$(7.5) \quad \dim \tilde{p}_h^{[0],-1} - \dim \nabla \cdot \tilde{V}_h^\circ = 1$$

for the choice (7.1), with  $p = 0$ , on this mesh  $\sum_h^2$ .

Now consider the case of piecewise quadratic fields, i.e. the choice  $\tilde{V}_h^\circ = (\tilde{p}_h^{[2],0})^2$  on the mesh  $\sum_h^1$ . The dimension of  $\tilde{p}_h^{[3],1} = p_h^{[3],1} \cap \{\psi : \psi = \frac{\partial \psi}{\partial n} = 0 \text{ on } \partial\Omega\}$  on a rectangle triangulated by this mesh has been determined in Chui, Schumaker & Wang [9], and via calculations similar to those just given we find that

$$(7.6) \quad \dim \tilde{p}_h^{[1],-1} - \dim \nabla \cdot \tilde{V}_h^\circ = 3 .$$

Note that this agrees with the results of Malkus & Olsen [22]. For the space  $\tilde{V}_h^\circ = (\tilde{p}_h^{[2],0})^2$  on  $\sum_h^2$  the situation is different. Mercier [23] observed that in this case  $\tilde{p}_h^{[3],1}$  has a local basis given by the Fraeijs de Veubeke-Sander cubic macro-element. Using calculations similar to those above one can show that

$$(7.7) \quad \nabla \cdot \tilde{V}_h^\circ = \tilde{p}_h^{[1],-1}$$

on a rectangle triangulated by the mesh  $\sum_h^2$  ( $p=1$ ).

For the case  $p = 2$ , that is piecewise

cubic fields,  $\mathring{V}_h = (\mathring{P}^{[3],0})^2$ , with homogeneous boundary conditions on a rectangle triangulated by the mesh  $\Sigma_h^2$ , one can use the local basis for  $\mathring{P}_h^{[4],1}$ , given via macro-elements in Douglas et al. [12], to show that

$$(7.8) \quad \nabla \cdot \mathring{V}_h = \tilde{P}_h^{[2],-1}.$$

To the best of our knowledge, no result is known for the mesh  $\Sigma_h^1$  for  $p = 2$ , although it seems clear that one would not expect (7.8) to hold in general.

Whenever, as in (7.5) and (7.6), the codimension of  $\nabla \cdot \mathring{V}_h$  in  $\tilde{P}_h^{[p],-1}$  is non-zero but independent of the mesh size, it is reasonable to conjecture that the orthogonal complements consist of global modes. For an explicit calculation of the global constraint on  $\nabla \cdot \mathring{V}_h$  in the case corresponding to (7.5), see Olsen [28].

The previous discussion centered on the characterization of  $\nabla \cdot V_h$  or  $\nabla \cdot \mathring{V}_h$  for low degree polynomials; we now turn to the aspects a) and c) listed at the beginning of this section. Proposition 5.1 shows that there is a close connection between approximation properties of the spaces  $\mathring{V}_h \cap \{\underline{v} : \nabla \cdot \underline{v} = 0\}$  (or  $V_h \cap \{\underline{v} : \nabla \cdot \underline{v} = 0\}$ ) and the divergence stability of  $\mathring{V}_h$  (or  $V_h$ , respectively).

In the following we shall for simplicity assume that  $\Omega$  is simply connected. Then  $V_h \cap \{\underline{v} : \nabla \cdot \underline{v} = 0\} = \nabla \times P_h^{[p+2],1}$  and  $\mathring{V}_h \cap \{\underline{v} : \nabla \cdot \underline{v} = 0\} = \nabla \times \mathring{P}_h^{[p+2],1}$ , and this in combination with Proposition 5.1 shows that there is a close connection between approximation properties of  $P_h^{[p+2],1}$  and  $\mathring{P}_h^{[p+2],1}$  and divergence-stability. The next proposition elaborates more on that connection.

**Proposition 7.1**

Let  $p = 0, 1$  or  $2$  and assume that there exist  $\psi \in H^{p+3}(\Omega)$ ,  $c > 0$ , and  $0 < \alpha \leq p + 3$  such that

$$\inf \|\psi - \phi\|_{L_2(\Omega)} \geq c h^{p+3-\alpha}$$

where the  $\inf$  is taken over  $\phi \in P_h^{[p+2],1}$  (for some quasi-uniform family of triangulations). Then

$$\inf_{q \in \nabla \cdot V_h \setminus \{0\}} \sup_{\underline{v} \in V_h \setminus \{0\}} \frac{b(\underline{v}, q)}{\|\underline{v}\|_{H^1(\Omega)} \|q\|_{L_2(\Omega)}} \leq C h^\beta$$

for the subspaces  $V_h = (P_h^{[p+1],0})^2$ , with  $\beta = \frac{p+1}{p+3} \alpha$ .

Proof:

By contradiction let us assume that

$$C_h = h^{-\beta} \inf_{q \in \nabla \cdot V_h \setminus \{0\}} \sup_{\underline{v} \in V_h \setminus \{0\}} \frac{b(\underline{v}, q)}{\|\underline{v}\|_{H^1(\Omega)} \|q\|_{L_2(\Omega)}}$$

is unbounded as  $h \rightarrow 0$ . Using the same argument as in the proof of Proposition 5.1 we get directly from the definition of  $C_h$  that

$$(7.9) \quad \inf_{\underline{z} \in V_h \cap \{\underline{v} : \nabla \cdot \underline{v} = 0\}} \|\underline{U} - \underline{z}\|_{H^1(\Omega)} \leq C C_h^{-1} h^{-\beta} \inf_{\underline{v} \in V_h} \|\underline{U} - \underline{v}\|_{H^1(\Omega)}$$

for any  $\underline{U} \in [H^1(\Omega)]^2 \cap \{\underline{v} : \nabla \cdot \underline{v} = 0\}$ . For any  $\psi \in H^{p+3}(\Omega)$  we get that  $\nabla \times \psi \in [H^{p+2}(\Omega)]^2 \cap \{\underline{v} : \nabla \cdot \underline{v} = 0\}$ , and by insertion into (7.9) it follows that

$$\begin{aligned} \inf_{\phi \in P_h^{[p+2],1}} \|\nabla \times \psi - \nabla \times \phi\|_{H^1(\Omega)} &\leq C C_h^{-1} h^{-\beta} \inf_{\underline{v} \in V_h} \|\nabla \times \psi - \underline{v}\|_{H^1(\Omega)} \\ &\leq C h^{p+1} C_h^{-1} h^{-\beta} \|\psi\|_{H^{p+3}(\Omega)}. \end{aligned}$$

From this we conclude that, for any  $\psi \in H^{p+3}(\Omega)$ ,

$$\inf_{\phi \in P_h^{[p+2],1}} \|\psi - \phi\|_{H^2(\Omega)} \leq C C_h^{-1} h^{p+1-\beta} \|\psi\|_{H^{p+3}(\Omega)}.$$

Using the results of Bramble and Scott [5] (these results are valid if we require that all corners of  $\Omega$  have interior angles  $< 2\pi$ ) we thus get

$$\inf_{\phi \in P_h^{[p+2],1}} \|\psi - \phi\|_{L^2(\Omega)} \leq C C_h^{-(p+3)/(p+1)} h^{p+3-\alpha} \|\psi\|_{H^{p+3}(\Omega)},$$

which due to the fact that  $C_h$  is unbounded as  $h \rightarrow 0$  produces a contradiction to the assumptions of this proposition.

Proposition 7.1 could just as easily have been verified for  $H^{p+3}(\Omega) \cap H^2(\Omega)$ ,  $P_h^{[p+2],1} = \{\phi \in P_h^{[p+2],1} : \phi = \frac{\partial \phi}{\partial n} = 0 \text{ at } \partial\Omega\}$  and  $\tilde{V}_h = (P_h^{[p+1],0})^2$ .

In the light of Propositions 5.1 and 7.1 we shall center our discussion of the aspects a) and c) on known results concerning the approximation rates of  $P_h^{[p+2],1}$  and  $\tilde{P}_h^{[p+2],1}$  on the meshes  $\Sigma_h^1$  and  $\Sigma_h^2$ .

First, suppose  $p = 0$ , the case of piecewise linear fields. On the mesh  $\Sigma_h^1$  we know as previously stated that

$$\tilde{P}_h^{[2],1} = \{0\};$$

this space obviously has no approximation rate associated to it and the assumptions of Proposition 7.1 are thus satisfied with  $\alpha = 3$ , i.e. the "best" constant in the "divergence-stability" estimate is  $\leq C h$ , cf. Gunzburger & Nicolaides [16]. We do not anticipate that relaxing the boundary conditions would lead to any approximation rates for the spaces  $P_h^{[2],1}$  on the mesh  $\Sigma_h^1$ , though we do not know this for a fact. On the mesh  $\Sigma_h^2$  Fix, Gunzburger & Nicolaides [13] give a construction of a uniformly bounded inverse for the divergence operator  $(\nabla \cdot)^{-1}: P_h^{[0],-1} \rightarrow V_h = [P_h^{[1],0}]^2$ , uniformly bounded that is in  $B(H^{-1}; L_2)$ , not  $B(L_2; H^1)$  as we are concerned with. Powell [30] and



Mercier [23] both show that  $p_h^{[2],1}$  has optimal approximation rates on this mesh. Malkus and Olsen [22] conjecture that optimal approximation properties hold as well for  $\overset{\circ}{p}_h^{[2],1}$  on  $\sum_h^2$ , although this has not been proven. (Because of the local basis for  $\overset{\circ}{p}_h^{[2],1}$  given by Powell [30], one would have  $\alpha = 1$  at worst.)

Now consider piecewise quadratic fields, i.e.  $p = 1$ . On the mesh  $\sum_h^1$  de Boor and Höllig [11] show that approximation by  $p_h^{[3],1}$  is suboptimal by precisely one order of  $h$  in  $L_\infty$ . This result (adapted to  $L_2$ ) according to Proposition 7.1 leads to the conclusion that the "best" constant in the "divergence-stability" estimate for the piecewise quadratic spaces  $V_h$  is  $\leq C h^{1/2}$ . Malkus and Olsen [22] report numerical evidence that the corresponding constant for the piecewise quadratic spaces  $\overset{\circ}{V}_h$  (with boundary conditions) is  $O(h)$  on the mesh  $\sum_h^1$ . On the mesh  $\sum_h^2$  the situation is quite different since both  $p_h^{[3],1}$  and  $\overset{\circ}{p}_h^{[3],1}$  are long known to have optimal approximation rates, cf. Mercier [23]; indeed, the recent results of Boland and Nicolaides [4] together with the result (7.7) show that the corresponding spaces  $V_h$  and  $\overset{\circ}{V}_h$  are divergence stable on the mesh  $\sum_h^2$  for  $p = 1$ . (For macroelement, one takes simply the quadrilateral surrounding each singular vertex. Local stability is then guaranteed by (7.7) applied to each macroelement. The reason that these macroelements are locally stable in the sense of Boland and Nicolaides [4] is that the boundary vertices of each macroelement are nonsingular.)

For piecewise cubic fields ( $p = 2$ ) on the mesh  $\sum_h^1$ , numerical experience (e.g. Szabo et al. [34]) had indicated that neither  $V_h$  nor  $\overset{\circ}{V}_h$  was divergence-stable. Moreover, the results of Jia [19], similar to those of de Boor and Höllig [11] quoted above, also lead to this conclusion via Proposition 7.1. For the mesh  $\sum_h^2$  and  $p = 2$ , the results of Douglas, et al. [12]

prove that (7.8) holds and consequently that both  $V_h$  and  $\dot{V}_h$  are divergence-stable, in the same fashion as for the case  $p = 1$  described previously. Finally, note that all the results for  $p = 1$  and 2 for the mesh  $\sum_h^2$  hold as well for a mesh based on the macroelement of Clough and Tocher, cf. Mercier [23].

To summarize the previous discussion:

1) The characterization of the range of the divergence operator on spaces of continuous piecewise polynomials of degree at most  $p + 1$ ,  $p \geq 3$ , given in section 6, is widely valid also for  $p \leq 2$ , provided no boundary conditions are imposed. With boundary conditions imposed this characterization fails on the most natural triangulations.

2) The divergence stability of the spaces  $[P^{[p+1],0}]^2$  or  $[\dot{P}^{[p+1],0}]^2$  is intimately connected to the approximation properties of the spaces  $P_h^{[p+2],1}$  or  $\dot{P}_h^{[p+2],1}$ . Since these spaces of  $C^1$  piecewise polynomials have essentially one additional degree of freedom for each singular vertex, it is only natural that the spaces  $[P^{[p+1],0}]^2$  or  $[\dot{P}^{[p+1],0}]^2$ ,  $p \leq 2$ , are much more likely to be divergence-stable, the more singular vertices the triangulation has. In this sense singular vertices are desirable when working with piecewise polynomials of degree  $\leq 3$ .

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